

Stochastic Differential Equations Driven by Fractional Brownian Motion and Standard Brownian Motion

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Abstract

We prove an existence and uniqueness theorem for solutions of multidimensional, time dependent, stochastic differential equations driven simultaneously by a multidimensional fractional Brownian motion with Hurst parameter $H > 1/2$ and a multidimensional standard Brownian motion. The proof relies on some a priori estimates, which are obtained using the methods of fractional integration, and the classical Itô stochastic calculus. The existence result is based on the Yamada-Watanabe theorem.

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1 Introduction

The fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with covariance function

$$R_H(s, t) = \frac{1}{2} \left(t^{2H} + s^{2H} - |t - s|^{2H} \right). \quad (1.1)$$

This process was introduced by Kolmogorov in [12] and later studied by Mandelbrot and Van Ness in [17]. Its self-similar and long-range dependence (if $H > \frac{1}{2}$) properties make this process a useful driving noise in models arising in physics, telecommunication networks, finance and other fields. For a review of some applications of fBm we refer to [6].

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From (1.1) we deduce that $\mathbb{E}(|B_t^H - B_s^H|^2) = |t-s|^{2H}$ and, as a consequence, the trajectories of B^H are almost surely locally α -Hölder continuous for all $\alpha \in (0, H)$. Since B^H is not a semimartingale if $H \neq 1/2$ (see [20]), we cannot use the classical Itô theory to construct a stochastic calculus with respect to the fBm. Over the last years some new techniques have been developed in order to define stochastic integrals with respect to fBm. Essentially two different types of integrals can be defined:

- i) The Skorokhod integral (or divergence integral) with respect to fBm is defined as the adjoint of the derivative operator in the framework of the Malliavin calculus. This approach was introduced by Decreusefond and Üstünel [5], and later developed by Carmona and Coutin [2], Duncan, Hu and Pasik-Duncan [7], Alòs, Mazet and Nualart [1], Hu and Øksendal [11], Cheridito and Nualart [3], among others. This stochastic integral satisfies the zero mean property and it can be expressed as the limit of Riemann sums defined using Wick products.
- ii) The pathwise Riemann-Stieltjes integral $\int_0^T v_s dB_s^H$ exists if $\{v_t, t \in [0, T]\}$ is a stochastic process with Hölder continuous paths of order $\alpha > 1 - H$, as a consequence of the results of Young [25]. Zähle expressed this integral in terms of fractional derivative operators, using the fractional integration by parts formula (see [26]). We also refer to [8] for a pathwise approach to the stochastic calculus based on the regularization of the noise.

The aim of this paper is to study the d -dimensional stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma_W(s, X_s) dW_s + \int_0^t \sigma_H(s, X_s) dB_s^H, \quad (1.2)$$

where W is an r -dimensional standard Brownian motion and B^H is an m -dimensional fractional Brownian motion with $H \in (\frac{1}{2}, 1)$. We assume that the processes W and B^H are independent. In the above stochastic differential equation, the integral $\int_0^t \sigma_W(s, X_s) dW_s$ should be interpreted as an Itô stochastic integral and the integral $\int_0^t \sigma_H(s, X_s) dB_s^H$ as a pathwise Riemann-Stieltjes integral in the sense of Zähle [26]. Our main result is a general theorem about the existence and uniqueness of solutions for the stochastic differential equation (1.2) under suitable conditions on the coefficients.

Equations driven only by a fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$ can be solved by a pathwise approach using the p -variation norm (see [15]), the fractional calculus introduced by Zähle (see [26] and [19]), or Hölder norms [21]. Also using the tools of rough path analysis introduced by Lyons in [16], Coutin and Qian proved in [4] the existence of strong solutions for stochastic differential equations driven by fBm with $H > \frac{1}{4}$ and studied a Wong-Zakai approximation limit for these stochastic differential equations.

Kubilius has studied stochastic differential equations driven by both fBm and standard Brownian motion (see [13]), in the one dimensional case, with σ_W, σ_H independent of the time and with no drift term ($b \equiv 0$). In this setting, the author proves an existence and uniqueness result provided that σ_W is a Lipschitz function and $\sigma_H \in C^{1+\delta}$, with $\delta > q(1-H)$, $q > 2$. With these assumptions, the solution belongs to the space of continuous functions with

q -bounded variation. Kubilius defines the stochastic integral with respect to fBm as an extended Riemann-Stieltjes pathwise integral and he uses p -variation estimates.

Our approach is completely different from Kubilius [13] in the sense that we combine the pathwise approach with the Itô stochastic calculus in order to handle both types of integrals. Then, the uniqueness of a solution follows from estimates for both Itô and Riemann-Stieltjes integrals. However, the existence of a strong solution cannot be obtained by the classical fixed point argument because the estimates of the Hölder norm of an integral with respect to B^H produce some higher order terms. For this reason, we first prove the existence of weak solutions and later deduce the existence of strong solutions using the Yamada-Watanabe theorem.

The paper is organized as follows. In Section 2 we state the problem and list our assumptions on the coefficients of Eq. (1.2). Section 3 provides some estimates for fractional and Itô integrals. In section 4, the pathwise uniqueness property of solutions of Eq. (1.2) is proved. In section 5, we introduce the Euler sequence that approximates the solution of the stochastic differential equation and prove that it is a tight sequence. Then, the Skorokhod representation theorem is applied in order to prove the existence of a weak solution for the stochastic differential equation. Finally, we prove the existence of a unique strong solution by using the Yamada-Watanabe theorem.

2 Preliminaries

Fix a time interval $[0, T]$ and a complete probability space (Ω, \mathcal{F}, P) . Suppose that $B^H = \{B_t^H, t \in [0, T]\}$ is an m -dimensional fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, and $W = \{W_t, t \in [0, T]\}$ an r -dimensional standard Brownian motion, independent of B^H . Let X_0 be a d -dimensional random variable independent of (B^H, W) . For each $t \in [0, T]$ we denote by \mathcal{F}_t the σ -field generated by the random variables $\{X_0, B_s^H, W_s, s \in [0, t]\}$ and the P -null sets.

In addition to the natural filtration $\{\mathcal{F}_t, t \in [0, T]\}$ we will consider bigger filtrations $\{\mathcal{G}_t, t \in [0, T]\}$ such that:

1. $\{\mathcal{G}_t\}$ is right-continuous and \mathcal{G}_0 contains the P -null sets.
2. X_0 and B^H are \mathcal{G}_0 -measurable, and W is a \mathcal{G}_t -Brownian motion.

Notice that $\widehat{\mathcal{F}}_t \subset \mathcal{G}_t$, where $\widehat{\mathcal{F}}_t$ is the σ -field generated by the random variables $\{X_0, B^H, W_s, s \in [0, t]\}$ and the P -null sets.

Consider the stochastic differential equation (1.2), where X_0 is a d -dimensional random variable independent of (B^H, W) and the coefficients are measurable functions $b^i, \sigma_W^{i,k}, \sigma_H^{i,j} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$, $1 \leq i \leq d$, $1 \leq k \leq r$, $1 \leq j \leq m$. We will make use of the following assumptions on the coefficients.

- (Hb) The function $b(t, x)$ is continuous. Moreover, it is Lipschitz continuous in the variable x and has linear growth in the same variable, uniformly in t , that is, there exist constants L_1 and L_2 such that

$$\begin{aligned} |b(t, x) - b(t, y)| &\leq L_1 |x - y|, \\ |b(t, x)| &\leq L_2 (1 + |x|), \end{aligned}$$

for all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$.

(H σ_W) The function $\sigma_W(t, x)$ is continuous. Moreover, it is Lipschitz continuous in x and has linear growth in the same variable, uniformly in t , that is, there exist constants L_3 and L_4 such that

$$\begin{aligned} |\sigma_W(t, x) - \sigma_W(t, y)| &\leq L_3 |x - y|, \\ |\sigma_W(t, x)| &\leq L_4 (1 + |x|), \end{aligned}$$

for all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$.

(H σ_H) The function $\sigma_H(t, x)$ is continuous and continuously differentiable in the variable x . Moreover, there exist constants L_5 , L_6 and L_7 such that

$$\begin{aligned} |\partial_{x_i} \sigma_H(t, x)| &\leq L_5, \\ |\partial_{x_i} \sigma_H(t, x) - \partial_{x_i} \sigma_H(t, y)| &\leq L_6 |x - y|^\delta, \\ |\sigma_H(t, x) - \sigma_H(s, x)| + |\partial_{x_i} \sigma_H(t, x) - \partial_{x_i} \sigma_H(s, x)| &\leq L_7 |t - s|^\beta, \end{aligned}$$

for all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$, and for some constants $0 < \delta, \beta \leq 1$.

Note that assumption (H σ_H) implies the linear growth property, i. e., there exists a constant L such that

$$|\sigma_H(t, x)| \leq L (1 + |x|) \quad (2.1)$$

for all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$.

Let us now introduce some function spaces that will be used in the analysis of solutions of the stochastic differential equation (1.2). Let $0 < \alpha < \frac{1}{2}$. For any measurable function $f : [0, T] \rightarrow \mathbb{R}^d$ we introduce the following notation

$$\|f(t)\|_\alpha := |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t - s)^{\alpha+1}} ds. \quad (2.2)$$

Denote by $W_0^{\alpha, \infty}$ the space of measurable functions $f : [0, T] \rightarrow \mathbb{R}^d$ such that

$$\|f\|_{\alpha, \infty} := \sup_{t \in [0, T]} \|f(t)\|_\alpha < \infty. \quad (2.3)$$

For $\mu \in (0, 1]$ let C^μ be space of μ -Hölder continuous functions $f : [0, T] \rightarrow \mathbb{R}^d$, equipped with the norm

$$\|f\|_\mu := \|f\|_\infty + \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t - s)^\mu} < \infty, \quad (2.4)$$

where $\|f\|_\infty := \sup_{t \in [0, T]} |f(t)|$. Given any ε such that $0 < \varepsilon < \alpha$, we have the following inclusions:

$$C^{\alpha+\varepsilon} \subset W_0^{\alpha, \infty} \subset C^{\alpha-\varepsilon}. \quad (2.5)$$

In particular, both the fractional Brownian motion B^H , with $H > \frac{1}{2}$, and the standard Brownian motion W , have their trajectories in $W_0^{\alpha, \infty}$. We denote by \mathbb{E}^W the conditional expectation given $\widehat{\mathcal{F}}_0$, that is, given X_0 and B^H .

We now define the space of processes where we will search for solutions of (1.2).

Definition 2.1 Let $\mathbb{W}_{\mathcal{G}}$ be the space of d -dimensional \mathcal{G}_t -adapted stochastic processes $X = \{X_t, t \in [0, T]\}$ such that almost surely the trajectories of X belong to $W_0^{\alpha, \infty}$ and $\int_0^T \mathbb{E}^W [\|X_s\|_{\alpha}^2] ds < \infty$.

A strong solution of the stochastic differential equation (1.2) is a stochastic process X in the space $\mathbb{W}_{\mathcal{F}}$, which satisfies (1.2) a.s. The main result proved in this paper is the following theorem on the uniqueness and existence of strong solutions for (1.2).

Theorem 2.2 Assume that the coefficients b , σ_W and σ_H satisfy the assumptions (Hb) , $(H\sigma_W)$ and $(H\sigma_H)$. If α satisfies $1 - H < \alpha < \min \{\frac{1}{2}, \beta, \frac{\delta}{2}\}$, then there exists a unique strong solution X of Equation (1.2).

Remark Notice that in all our results we can replace the fractional Brownian motion B^H by an arbitrary stochastic process with Hölder continuous trajectories of order $\gamma > \frac{1}{2}$.

3 Integral estimates

In this section we will first define the integral with respect to fBm as a generalized Stieltjes integral, following the work of Zähle [26]. We also present some basic estimates of this integral.

Let $f \in L^1(a, b)$ and $\alpha > 0$. The left-sided and right-sided fractional Riemann-Liouville integrals of f of order α are defined for almost all $x \in (a, b)$ by

$$I_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy$$

and

$$I_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy$$

respectively, where $\Gamma(\alpha) := \int_0^{\infty} r^{\alpha-1} e^{-r} dr$ is the Euler gamma function. Let $I_{a+}^{\alpha}(L^p)$ (resp. $I_{b-}^{\alpha}(L^p)$) be the image of $L^p(a, b)$ by the operator I_{a+}^{α} (resp. I_{b-}^{α}). If $f \in I_{a+}^{\alpha}(L^p)$ (resp. $f \in I_{b-}^{\alpha}(L^p)$) and $0 < \alpha < 1$ then the Weyl derivatives of f are given by

$$D_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right) 1_{(a,b)}(x) \quad (3.1)$$

and

$$D_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}} dy \right) 1_{(a,b)}(x), \quad (3.2)$$

respectively, and are defined for almost all $x \in (a, b)$ (the convergence of the integrals at the singularity $y = x$ holds pointwise for almost all $x \in (a, b)$ if $p = 1$ and moreover in L^p -sense if $1 < p < \infty$).

We have that:

- If $\alpha < \frac{1}{p}$ and $q = \frac{p}{1-\alpha p}$ then $I_{a+}^{\alpha}(L^p) = I_{b-}^{\alpha}(L^p) \subset L^q(a, b)$.

- If $\alpha > \frac{1}{p}$ then $I_{a+}^\alpha(L^p) \cup I_{b-}^\alpha(L^p) \subset C^{\alpha-\frac{1}{p}}(a, b)$.

The fractional integrals and derivatives are related by the inversion formulas

$$I_{a+}^\alpha(D_{a+}^\alpha f) = f, \quad \forall f \in I_{a+}^\alpha(L^p),$$

$$D_{a+}^\alpha(I_{a+}^\alpha f) = f, \quad \forall f \in L^1(a, b),$$

and similar formulas also hold for I_{b-}^α and D_{b-}^α . We refer to [22] for a detailed account on the properties of fractional operators.

Let $f(a+) := \lim_{\varepsilon \searrow 0} f(a + \varepsilon)$ and $g(b-) := \lim_{\varepsilon \searrow 0} g(b - \varepsilon)$ (we are assuming that these limits exist and are finite) and define

$$f_{a+}(x) := (f(x) - f(a+)) \mathbf{1}_{(a,b)}(x),$$

$$g_{b-}(x) := (g(x) - g(b-)) \mathbf{1}_{(a,b)}(x).$$

We recall from [26] the definition of generalized Stieltjes fractional integral with respect to irregular functions.

Definition 3.1 (Generalized Stieltjes Integral) Suppose that f and g are functions such that $f(a+), g(a+)$ and $g(b-)$ exist, $f_{a+} \in I_{a+}^\alpha(L^p)$ and $g_{b-} \in I_{b-}^{1-\alpha}(L^p)$ for some $p, q \geq 1, 1/p + 1/q \leq 1, 0 < \alpha < 1$. Then the integral of f with respect to g is defined by

$$\int_a^b f dg := (-1)^\alpha \int_a^b D_{a+}^\alpha f(x) D_{b-}^{1-\alpha} g_{b-}(x) dx + f(a+) (g(b-) - g(a+)).$$

Remark 3.2 The above definition is simpler in the following cases.

- If $\alpha p < 1$, under the assumptions of the preceding definition, we have that $f \in I_{a+}^\alpha(L^p)$ and we can write

$$\int_a^b f dg = (-1)^\alpha \int_a^b D_{a+}^\alpha f_{a+}(x) D_{b-}^{1-\alpha} g_{b-}(x) dx. \quad (3.3)$$

- If $f \in C^\lambda(a, b)$ and $g \in C^\mu(a, b)$ with $\lambda + \mu > 1$ then (see [26]) we can choose α such that $1 - \mu < \alpha < \lambda$, the generalized Stieltjes integral exists, it is given by (3.3) and coincides with the Riemann-Stieltjes integral.

The linear spaces $I_{a+}^\alpha(L^p)$ are Banach spaces with respect to the norms

$$\|f\|_{I_{a+}^\alpha(L^p)} := \|f\|_{L^p} + \|D_{a+}^\alpha f\|_{L^p} \sim \|D_{a+}^\alpha f\|_{L^p},$$

and the same is true for the spaces $I_{b-}^\alpha(L^p)$. If $\alpha p < 1$ then the norms on $I_{a+}^\alpha(L^p)$ and $I_{b-}^\alpha(L^p)$ are equivalent and if $a \leq c < d \leq b$, then

$$\int_c^d f dg := \int_a^b \mathbf{1}_{(c,d)} f dg.$$

Now, fix the parameter α such that $0 < \alpha < \frac{1}{2}$, denote by $W_T^{1-\alpha, \infty}$ the space of measurable functions $g : [0, T] \rightarrow \mathbb{R}^m$ such that

$$\|g\|_{1-\alpha, \infty, T} := \sup_{0 \leq s < t < T} \left(\frac{|g(t) - g(s)|}{(t - s)^{1-\alpha}} + \int_s^t \frac{|g(y) - g(s)|}{(y - s)^{2-\alpha}} dy \right) < \infty.$$

and denote by $W_0^{\alpha,1}$ the space of measurable functions $f : [0, T] \rightarrow \mathbb{R}^d$ such that

$$\|f\|_{\alpha,1} := \int_0^T \frac{|f(s)|}{s^\alpha} ds + \int_0^T \int_0^s \frac{|f(s) - f(y)|}{(s-y)^{\alpha+1}} dy ds < \infty.$$

It is easy to prove that

$$C^{1-\alpha+\varepsilon} \subset W_T^{1-\alpha,\infty} \subset C^{1-\alpha},$$

for all $\varepsilon > 0$. For $g \in W_T^{1-\alpha,\infty}$, we have that

$$\begin{aligned} \Lambda_\alpha(g) &:= \frac{1}{\Gamma(1-\alpha)} \sup_{0 < s < t < T} |(D_{t-}^{1-\alpha} g_{t-})(s)| \\ &\leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha,\infty,T} < \infty. \end{aligned}$$

Moreover, if $f \in W_0^{\alpha,1}$ and $g \in W_T^{1-\alpha,\infty}$ then $\int_0^t f dg$ exists for all $t \in [0, T]$ and

$$\left| \int_0^t f dg \right| \leq \Lambda_\alpha(g) \|f\|_{\alpha,1}. \quad (3.4)$$

Now we will deduce useful estimates for the integrals involved in Equation (1.2). Fix $\alpha \in (1-H, \frac{1}{2})$. We will denote by C a generic constant which depends on the constants L_i , $1 \leq i \leq 7$, β and δ in the assumptions, on T , α and the dimensions r, d, m . For any function $f \in W_0^{\alpha,\infty}$ define

$$F_t^b(f) := \int_0^t b(s, f(s)) ds.$$

Proposition 3.3 *If $f \in W_0^{\alpha,\infty}$ then $F^b(f) \in W_0^{\alpha,\infty}$ and for all $t \in [0, T]$*

$$\|F_t^b(f)\|_\alpha \leq C \left(\int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds + 1 \right). \quad (3.5)$$

Proof. By Proposition 4.3 in [19] and the growth assumption in (Hb) we have that

$$\begin{aligned} \|F_t^b(f)\|_\alpha &\leq C \int_0^t \frac{|b(s, f(s))|}{(t-s)^\alpha} ds \\ &\leq C \int_0^t \frac{|f(s)| + 1}{(t-s)^\alpha} ds \\ &\leq C \left(\int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds + 1 \right). \end{aligned}$$

■

Proposition 3.4 *If $f, h \in W_0^{\alpha,\infty}$ then for all $t \in [0, T]$*

$$\|F_t^b(f) - F_t^b(h)\|_\alpha \leq C \int_0^t \frac{\|f(s) - h(s)\|_\alpha}{(t-s)^\alpha} ds. \quad (3.6)$$

Proof. By Proposition 4.3 in [19] and the Lipschitz assumption in (Hb), we have that

$$\begin{aligned}\|F_t^b(f) - F_t^b(h)\|_\alpha &\leq C \int_0^t \frac{|b(s, f(s)) - b(s, h(s))|}{(t-s)^\alpha} ds \\ &\leq C \int_0^t \frac{|f(s) - h(s)|}{(t-s)^\alpha} ds.\end{aligned}$$

■

Given a function $f \in W_0^{\alpha, \infty}$, let us define

$$G_t^{\sigma H}(f) := \int_0^t \sigma_H(s, f(s)) dB_s^H.$$

Proposition 3.5 Suppose $1 - H < \alpha < \min(\frac{1}{2}, \beta)$. Then for all $t \in [0, T]$

$$\|G_t^{\sigma H}(f)\|_\alpha \leq C \Lambda_\alpha(B^H) \int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) (1 + \|f(s)\|_\alpha) ds. \quad (3.7)$$

Proof. By Proposition 4.1 of [19] and the Hölder continuity in time, given in assumption $(H\sigma_H)$, we have

$$\begin{aligned}\|G_t^{\sigma H}(f)\|_\alpha &\leq C \Lambda_\alpha(B^H) \int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) \|\sigma_H(s, f(s))\|_\alpha ds \\ &\leq C \Lambda_\alpha(B^H) \int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) (1 + \|f(s)\|_\alpha) ds.\end{aligned}$$

■

Proposition 3.6 If $1 - H < \alpha < \min(\frac{1}{2}, \beta)$ and $f, h \in W_0^{\alpha, \infty}$ then for all $t \in [0, T]$

$$\begin{aligned}\|G_t^{\sigma H}(f) - G_t^{\sigma H}(h)\|_\alpha &\leq C \Lambda_\alpha(B^H) \\ &\times \int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) (1 + \Delta f(s) + \Delta h(s)) \|f(s) - h(s)\|_\alpha ds,\end{aligned} \quad (3.8)$$

where we denote

$$\Delta f(s) := \int_0^s \frac{|f(s) - f(r)|^\delta}{(s-r)^{\alpha+1}} dr. \quad (3.9)$$

Proof. From Proposition 4.1. of [19], we have that

$$\begin{aligned}\|G_t^{\sigma H}(f) - G_t^{\sigma H}(h)\|_\alpha &\leq C \Lambda_\alpha(B^H) \int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) \\ &\times \|\sigma_H(s, f(s)) - \sigma_H(s, h(s))\|_\alpha ds \\ &\leq C \Lambda_\alpha(B^H) \int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) (|\sigma_H(s, f(s)) - \sigma_H(s, h(s))| \\ &+ \int_0^s \frac{|\sigma_H(s, f(s)) - \sigma_H(s, h(s)) - \sigma_H(r, f(r)) + \sigma_H(r, h(r))|}{(s-r)^{\alpha+1}} dr) ds.\end{aligned}$$

Now, using the assumptions in $(H\sigma_H)$ and Lemma 7.1 in [19], we have that

$$\begin{aligned} & |\sigma(t_1, x_1) - \sigma(t_2, x_2) - \sigma(t_1, x_3) + \sigma(t_2, x_4)| \leq \\ & \leq L_5 |x_1 - x_2 - x_3 + x_4| + L_7 |x_1 - x_3| |t_2 - t_1|^\beta \\ & + L_6 |x_1 - x_3| \left(|x_1 - x_2|^\delta + |x_3 - x_4|^\delta \right). \end{aligned}$$

As a consequence,

$$\begin{aligned} & \|G_t^{\sigma H}(f) - G_t^{\sigma H}(h)\|_\alpha \leq C\Lambda_\alpha(B^H) \int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) \\ & \times (1 + \Delta f(s) + \Delta h(s)) \left(|f(s) - h(s)| + \int_0^s \frac{|f(s) - h(s) - f(r) + h(r)|}{(s-r)^{\alpha+1}} dr \right) ds \\ & \leq C\Lambda_\alpha(B^H) \int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) \\ & \times \left(1 + (\Delta f(s))^2 + (\Delta h(s))^2 \right) \|f(s) - h(s)\|_\alpha^2 ds. \end{aligned}$$

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Finally, we will consider the Itô stochastic integral with respect to the r -dimensional standard Brownian motion W . The following lemma is an immediate consequence of Itô calculus.

Lemma 3.7 *Suppose that $u = \{u(t), t \in [0, T]\}$ is an r -dimensional \mathcal{G}_t -adapted stochastic process such that $\int_0^T \mathbb{E}^W [u(s)^2] ds < \infty$. Then for all $t \in [0, T]$ a.e.*

$$\mathbb{E}^W \left[\left\| \int_0^t u(s) dW_s \right\|_\alpha^2 \right] \leq C \int_0^t (t-s)^{-\frac{1}{2}-\alpha} \mathbb{E}^W [u(s)^2] ds. \quad (3.10)$$

Proof. Notice first that, by Fubini's theorem, the right-hand side of (3.10) is finite for all $t \in [0, T]$ a.e. Applying the Itô isometry property and the Cauchy-Schwarz inequality, we have that:

$$\begin{aligned} & \mathbb{E}^W \left[\left\| \int_0^t u(s) dW_s \right\|_\alpha^2 \right] \leq C \mathbb{E}^W \left[\left| \int_0^t u(s) dW_s \right|^2 + \left(\int_0^t \frac{\left| \int_s^t u(r) dW_r \right|}{(t-s)^{\alpha+1}} ds \right)^2 \right] \\ & \leq C \left[\int_0^t \mathbb{E}^W [u(s)^2] ds + \frac{T^{\frac{1}{2}-\alpha}}{\frac{1}{2}-\alpha} \int_0^t \frac{\mathbb{E}^W \left| \int_s^t u(r) dW_r \right|^2}{(t-s)^{\frac{3}{2}+\alpha}} ds \right]. \end{aligned}$$

Therefore, by the Itô isometry and Fubini's theorem we obtain

$$\begin{aligned} & \mathbb{E}^W \left[\left\| \int_0^t u(s) dW_s \right\|_\alpha^2 \right] \\ & \leq C \left[\int_0^t \mathbb{E}^W [u(s)^2] ds + \frac{T^{\frac{1}{2}-\alpha}}{(\frac{1}{2}-\alpha)} \int_0^t \left[\int_0^r (t-s)^{-\frac{3}{2}-\alpha} ds \right] \mathbb{E}^W [u(r)^2] dr \right] \\ & \leq C \left[\int_0^t \mathbb{E}^W [u(s)^2] ds + \frac{T^{\frac{1}{2}-\alpha}}{(\frac{1}{2}-\alpha)(\frac{1}{2}+\alpha)} \int_0^t \frac{\mathbb{E}^W [u(r)^2]}{(t-r)^{\frac{1}{2}+\alpha}} dr \right] \\ & \leq C \int_0^t (t-r)^{-\frac{1}{2}-\alpha} \mathbb{E}^W [u(r)^2] dr. \end{aligned}$$

■

Assume that $f = \{f(t), t \in [0, T]\}$ is a d -dimensional stochastic process in $\mathbb{W}_{\mathcal{G}}$. Define

$$G_t^{\sigma W}(f) := \int_0^t \sigma(s, f(s)) dW_s.$$

We have the following estimates for these integrals:

Proposition 3.8 *Let $f \in \mathbb{W}_{\mathcal{G}}$. Then for all $t \in [0, T]$ a.e.*

$$\mathbb{E}^W \left[\|G_t^{\sigma W}(f)\|_{\alpha}^2 \right] \leq C \int_0^t (t-s)^{-\frac{1}{2}-\alpha} \left[1 + \mathbb{E}^W \left[\|f(s)\|_{\alpha}^2 \right] \right] ds. \quad (3.11)$$

Proof. It follows from (3.10) and the linear growth assumption in $(H\sigma_W)$.

■

Proposition 3.9 *Let $f, h \in \mathbb{W}_{\mathcal{G}}$. Then for all $t \in [0, T]$ a.e.*

$$\mathbb{E}^W \left[\|G_t^{\sigma W}(f) - G_t^{\sigma W}(h)\|_{\alpha}^2 \right] \leq C \int_0^t (t-s)^{-\frac{1}{2}-\alpha} \mathbb{E}^W \left[|f(s) - h(s)|^2 \right] ds. \quad (3.12)$$

Proof. By estimate (3.10) and the Lipschitz assumption in $(H\sigma_W)$, we obtain

$$\begin{aligned} & \mathbb{E}^W \left[\|G_t^{\sigma W}(f) - G_t^{\sigma W}(h)\|_{\alpha}^2 \right] \\ & \leq C \int_0^t (t-s)^{-\frac{1}{2}-\alpha} \mathbb{E}^W \left[|\sigma_W(s, f(s)) - \sigma_W(s, h(s))|^2 \right] ds \\ & \leq C \int_0^t (t-s)^{-\frac{1}{2}-\alpha} \mathbb{E}^W \left[|f(s) - h(s)|^2 \right] ds. \end{aligned}$$

■

4 Pathwise uniqueness

In this section we define the notion of weak solution for the stochastic differential equation (1.2) and we discuss the pathwise uniqueness of a solution.

Definition 4.1 *A weak solution of the stochastic differential equation (1.2) is a triple (X, B^H, W) , (Ω, \mathcal{F}, P) , $\{\mathcal{G}_t, t \in [0, T]\}$, where*

1. (Ω, \mathcal{F}, P) is a complete probability space, $\{\mathcal{G}_t\}$ is a right-continuous filtration such that \mathcal{G}_0 contains the P -null sets.
2. W is a \mathcal{G}_t - r -dimensional Brownian motion.
3. B^H is a fractional Brownian motion of Hurst parameter H which is \mathcal{G}_0 -measurable.
4. The process X is \mathcal{G}_t -adapted, has trajectories in $W_0^{\alpha, \infty}$ almost surely, and $\int_0^T \mathbb{E}^W \left[\|X_s\|_{\alpha}^2 \right] ds < \infty$ a.s.

5. (X, B^H, W) satisfies Equation (1.2) a.s.

Definition 4.2 We say that pathwise uniqueness holds for Equation (1.2) if, whenever (X, W, B^H) and (Y, W, B^H) are two weak solutions, defined on the same probability space (Ω, \mathcal{F}, P) with the same filtration $\{\mathcal{G}_t\}$ and $X_0 = Y_0$ a.s., then $X = Y$.

We will make use of the following technical lemma.

Lemma 4.3 Let $0 < \eta < 1/2$. If f is a continuous function such that $\|f\|_\eta \leq N$ and $\alpha < \eta\delta$, then $\Delta(f)$ is bounded by a constant C depending on T, N, α, δ , and η , where we use the notation introduced in (2.4) and (3.9).

Proof. Clearly

$$\Delta(f)(s) = \int_0^s \frac{|f(s) - f(r)|^\delta}{(s-r)^{\alpha+1}} dr \leq N^\delta \frac{T^{\eta\delta-\alpha}}{\eta\delta-\alpha},$$

which gives the result. \blacksquare

Let $f \in W_0^{\alpha, \infty}$. By the estimates proved in Proposition 4.2 and Proposition 4.4 of [19], the sample paths of the integral processes $F^b(f)$ and $G^{\sigma H}(f)$ are continuously differentiable and η -Hölder continuous of order $\eta < 1 - \alpha$, respectively. Therefore, if X is a weak solution of (1.2), then the trajectories of X are η -Hölder continuous for all $\eta < 1/2$.

Theorem 4.4 (Pathwise uniqueness) Let $1 - H < \alpha < \min\{\beta, \frac{\delta}{2}, \frac{1}{2}\}$. Then, the pathwise uniqueness property holds for Equation (1.2).

Proof. Let X and Y be two weak solutions of (1.2) defined on the same probability space, adapted to the same filtration and with the same initial value. Then the trajectories of X and Y are η -Hölder continuous, for all $\eta < 1/2$. Choose η such that $\alpha < \eta < 1/2$. Consider the sets $\Omega_N \subset \Omega$, defined by

$$\Omega_N := \left\{ \omega \in \Omega : \|X\|_\eta \leq N \text{ and } \|Y\|_\eta \leq N \right\},$$

with $N \in \mathbb{N}$. It is clear that $\Omega_N \nearrow \Omega$. From (1.2) we have that the difference between the two solutions satisfies

$$\begin{aligned} & \mathbb{E}^W \left[\|X_t - Y_t\|_\alpha^2 \mathbf{1}_{\Omega_N} \right] \\ & \leq 4\mathbb{E}^W \left\| (F_t^b(X) - F_t^b(Y)) \mathbf{1}_{\Omega_N} \right\|_\alpha^2 + 4\mathbb{E}^W \left\| (G_t^{\sigma W}(X) - G_t^{\sigma W}(Y)) \right\|_\alpha^2 \\ & \quad + 4\mathbb{E}^W \left\| (G_t^{\sigma H}(X) - G_t^{\sigma H}(Y)) \mathbf{1}_{\Omega_N} \right\|_\alpha^2. \end{aligned} \quad (4.1)$$

We split the set Ω into Ω_N and $\Omega \setminus \Omega_N$ in the second summand of (4.1) and use the estimates (3.6), (3.8), (3.12) in order to obtain

$$\begin{aligned} & \mathbb{E}^W \left[\|X_t - Y_t\|_\alpha^2 \mathbf{1}_{\Omega_N} \right] \\ & \leq C \int_0^t \varphi(s, t) \mathbb{E}^W \left[\|X_s - Y_s\|_\alpha^2 \mathbf{1}_{\Omega_N} \right] ds \\ & \quad + C \int_0^t \varphi(s, t) \left(\mathbb{E}^W \left[\|X_s - Y_s\|_\alpha^2 \mathbf{1}_{\Omega_N} \right] + \mathbb{E}^W \left[\|X_s - Y_s\|_\alpha^2 \mathbf{1}_{\Omega \setminus \Omega_N} \right] \right) ds \\ & \quad + C (\Lambda_\alpha(B^H))^2 \int_0^t \varphi(s, t) \mathbb{E}^W \left[(1 + (\Delta X_s)^2 + (\Delta Y_s)^2) \|X_s - Y_s\|_\alpha^2 \mathbf{1}_{\Omega_N} \right] ds, \end{aligned} \quad (4.2)$$

where

$$\varphi(s, t) = (t - s)^{-\frac{1}{2} - \alpha} + s^{-\alpha}.$$

If $\omega \in \Omega_N$ then, by Lemma 4.3, we have that

$$1 + (\Delta X_s)^2 + (\Delta Y_s)^2 \leq C_N. \quad (4.3)$$

Set

$$V_N(t) = \int_0^t \varphi(s, t) \mathbb{E}^W \left[\|X_s - Y_s\|_\alpha^2 \mathbf{1}_{\Omega_N} \right] ds.$$

Multiplying Equation (4.2) by $\varphi(s, t)$ and integrating, yields

$$\begin{aligned} V_N(t) &\leq C_N \left[(\Lambda_\alpha(B^H))^2 + 1 \right] \int_0^t \varphi(s, t) V_N(s) ds \\ &\quad + C \int_0^t \varphi(s, t) \int_0^s \varphi(r, s) \mathbb{E}^W \left[\|X_r - Y_r\|_\alpha^2 \mathbf{1}_{\Omega \setminus \Omega_N} \right] dr ds. \end{aligned} \quad (4.4)$$

By the bounded convergence theorem, we have that almost surely

$$V_N(t) \rightarrow \int_0^t \varphi(s, t) \mathbb{E}^W \left[\|X_s - Y_s\|_\alpha^2 \right] ds < \infty$$

and

$$\int_0^t \varphi(s, t) \int_0^s \varphi(r, s) \mathbb{E}^W \left[\|X_r - Y_r\|_\alpha^2 \mathbf{1}_{\Omega \setminus \Omega_N} \right] dr ds \rightarrow 0,$$

as N tends to infinity. Then, there exists a random variable $N^* \in \mathbb{N}$ such that

$$C \int_0^t \varphi(s, t) \int_0^s \varphi(r, s) \mathbb{E}^W \left[\|X_r - Y_r\|_\alpha^2 \mathbf{1}_{\Omega \setminus \Omega_N} \right] dr ds \leq \frac{1}{2} V_N(t), \quad (4.5)$$

for all $N \geq N^*$. Substituting (4.5) into (4.4) yields

$$V_N(t) \leq C_N \left[(\Lambda_\alpha(B^H))^2 + 1 \right] \int_0^t \varphi(s, t) V_N(s) ds,$$

for all $N \geq N^*$. Applying now the Gronwall-type Lemma 7.6 in [19], we deduce that $V_N(t) = 0$ for all $N \geq N^*$ almost surely. Hence,

$$P[X_t = Y_t, \quad \forall t \in [0, T]] = 1,$$

and the pathwise uniqueness property holds. \blacksquare

5 Existence of solutions

Let us now introduce the Euler approximations for Equation (1.2). Consider the framework (Ω, \mathcal{F}, P) , $\{\mathcal{F}_t, t \in [0, T]\}$, (X_0, B^H, W) introduced in Section 2. Fix a sequence of partitions

$$0 = t_0^n < t_1^n < \dots < t_i^n < \dots < t_n^n = T$$

of $[0, T]$ such that

$$\sup_{0 \leq i \leq n-1} |t_{i+1}^n - t_i^n| \rightarrow 0$$

as $n \rightarrow \infty$. Define $X^0(t) = X_0$ and for $n \geq 1$,

$$\begin{aligned} X^n(t) &= X_0 + \int_0^t b(k_n(s), X^n(k_n(s)))ds + \int_0^t \sigma_W(k_n(s), X^n(k_n(s)))dW_s \\ &\quad + \int_0^t \sigma_H(k_n(s), X^n(k_n(s)))dB_s^H, \end{aligned} \quad (5.1)$$

where

$$k_n(t) := t_i^n,$$

if $t \in [t_i^n, t_{i+1}^n)$. We will show the following result.

Proposition 5.1 *For any integer $N \geq 1$ there exists a random variable $R_N > 0$, depending on X_0 and B^H , such that, almost surely,*

$$\mathbb{E}^W \left[|X_t^n - X_s^n|^{2N} \right] \leq R_N |t - s|^N, \quad (5.2)$$

for all $s, t \in [0, T]$ and $n \in \mathbb{N}$.

Proof. The proof will be done in two steps.

Step 1.- We begin by proving that there is a random variable $K_N > 0$ such that

$$\mathbb{E}^W \left[\|X_t^n\|_\alpha^{2N} \right] \leq K_N, \quad (5.3)$$

for all $t \in [0, T]$ and for all $N \in \mathbb{N}$.

Note that the paths of $X^n(k_n(\cdot))$ are piecewise constant and the integrals in (5.1) are just finite sums. In the following computations, C_N denotes a positive constant that depends on N and the other parameters of the problem, and may vary from line to line. From (5.1), we have that

$$\begin{aligned} \mathbb{E}^W \left[\|X_t^n\|_\alpha^{2N} \right] &\leq C_N \left\{ |X_0|^{2N} + \mathbb{E}^W \left[\left\| \int_0^t b(k_n(s), X^n(k_n(s)))ds \right\|_\alpha^{2N} \right] \right. \\ &\quad + \mathbb{E}^W \left[\left\| \int_0^t \sigma_W(k_n(s), X^n(k_n(s)))dW_s \right\|_\alpha^{2N} \right] \\ &\quad \left. + \mathbb{E}^W \left[\left\| \int_0^t \sigma_H(k_n(s), X^n(k_n(s)))dB_s^H \right\|_\alpha^{2N} \right] \right\} \\ &= C_N \left(|X_0|^{2N} + A_1 + A_2 + A_3 \right). \end{aligned}$$

Using the estimate (3.5) and Hölder's inequality, we obtain

$$\begin{aligned} A_1 &\leq C_N \mathbb{E}^W \left[\left(\int_0^t \frac{|X^n(k_n(s))|}{(t-s)^\alpha} ds + 1 \right)^{2N} \right] \\ &\leq C_N \mathbb{E}^W \left[\left(\int_0^t |X^n(k_n(s))|^2 ds \right)^N \right] + C_N \\ &\leq C_N \mathbb{E}^W \left[\int_0^t |X^n(k_n(s))|^{2N} ds \right] + C_N. \end{aligned}$$

We have also that

$$\begin{aligned}
A_2 &\leq C_N \mathbb{E}^W \left[\left| \int_0^t \sigma_W(k_n(s), X^n(k_n(s))) dW_s \right|^{2N} \right] \\
&+ C_N \mathbb{E}^W \left[\left(\int_0^t \frac{\left| \int_s^t \sigma_W(k_n(r), X^n(k_n(r))) dW_r \right|^{2N}}{(t-s)^{\alpha+1}} ds \right)^{2N} \right] \\
&= A_{11} + A_{12}.
\end{aligned}$$

Applying Burkholder and Hölder inequalities, we have that

$$\begin{aligned}
A_{11} &\leq C_N \mathbb{E}^W \left[\left| \int_0^t |\sigma_W(k_n(s), X^n(k_n(s)))|^{2N} ds \right| \right] \\
&\leq C_N \int_0^t \left(1 + \mathbb{E}^W \left[|X^n(k_n(s))|^{2N} \right] \right) ds,
\end{aligned}$$

where we have used the linear growth assumption in $(H\sigma_W)$. For the second term we have, by Hölder and Burkholder inequalities, that

$$\begin{aligned}
A_{12} &\leq C_N \mathbb{E}^W \left[\left(\int_0^t \frac{1}{(t-s)^{\frac{2N}{2N-1}(\alpha+\frac{1}{2}-\frac{1/2+\alpha}{2N})}} ds \right)^{2N-1} \right. \\
&\quad \times \left. \left(\int_0^t \frac{\left| \int_s^t \sigma_W(k_n(r), X^n(k_n(r))) dW_r \right|^{2N}}{(t-s)^{N+\frac{1}{2}+\alpha}} ds \right) \right] \\
&\leq C_N \int_0^t (t-s)^{-\frac{3}{2}-\alpha} \mathbb{E}^W \left[\int_s^t |\sigma_W(k_n(r), X^n(k_n(r)))|^{2N} dr \right] ds.
\end{aligned}$$

Applying now Fubini's theorem and using the growth assumption in $(H\sigma_W)$, we obtain

$$A_{12} \leq C_N \int_0^t (t-r)^{-\frac{1}{2}-\alpha} \left(1 + \mathbb{E}^W \left[|X^n(k_n(r))|^{2N} \right] \right) dr.$$

Therefore,

$$A_2 \leq C_N \int_0^t (t-s)^{-\frac{1}{2}-\alpha} \mathbb{E}^W \left[|X^n(k_n(s))|^{2N} \right] ds + C_N.$$

Applying (3.7), we have that

$$A_3 \leq C_N \Lambda_\alpha(B^H)^{2N} \mathbb{E}^W \left[\left(\int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) \|\sigma_H(k_n(s), X^n(k_n(s)))\|_\alpha ds \right)^{2N} \right].$$

By Hölder's inequality and the assumptions in $(H\sigma_H)$, we have

$$A_3 \leq C_N \Lambda_\alpha(B^H)^{2N} \int_0^t ((t-s)^{-2\alpha} + s^{-\alpha}) \left[1 + \mathbb{E}^W \left[\|X^n(k_n(s))\|_\alpha^{2N} \right] \right] ds.$$

Putting together all the estimates obtained for A_1, A_2 and A_3 , we obtain

$$\begin{aligned} \mathbb{E}^W \left[\|X_t^n\|_\alpha^{2N} \right] &\leq C_N |X_0|^{2N} + C_N [\Lambda_\alpha(B^H)^{2N} + 1] \\ &\times \int_0^t \left((t-s)^{-\frac{1}{2}-\alpha} + s^{-\alpha} \right) \mathbb{E}^W \left[\|X^n(k_n(s))\|_\alpha^{2N} \right] ds. \end{aligned} \quad (5.4)$$

Therefore, since the right-hand side of Equation (5.4) is an increasing function of t , we have

$$\begin{aligned} \sup_{0 \leq s \leq t} \mathbb{E}^W \left[\|X_s^n\|_\alpha^{2N} \right] &\leq C_N |X_0|^{2N} + C_N [\Lambda_\alpha(B^H)^{2N} + 1] \\ &\times \int_0^t \left((t-s)^{-\frac{1}{2}-\alpha} + s^{-\alpha} \right) \left(\sup_{0 \leq u \leq s} \mathbb{E}^W \left[\|X_u^n\|_\alpha^{2N} \right] \right) ds. \end{aligned}$$

As a consequence, by the Gronwall-type lemma (Lemma 7.6 in [19]), we deduce the desired estimate.

Step 2.- Now we show that there is a random variable R_N such that (5.2) holds. In the sequel, R_N denotes a positive random variable. We have

$$\begin{aligned} \mathbb{E}^W \left[|X_t^n - X_s^n|^{2N} \right] &\leq C_N \left\{ \mathbb{E}^W \left[\left| \int_s^t b(k_n(u), X^n(k_n(u))) du \right|^{2N} \right] \right. \\ &+ \mathbb{E}^W \left[\left| \int_s^t \sigma_W(k_n(u), X^n(k_n(u))) dW_u \right|^{2N} \right] \\ &+ \mathbb{E}^W \left[\left| \int_s^t \sigma_H(k_n(u), X^n(k_n(u))) dB_u^H \right|^{2N} \right] \Big\} \\ &= B_1 + B_2 + B_3. \end{aligned}$$

Applying Hölder's inequality, the growth assumption in (Hb) and (5.3), we have that

$$\begin{aligned} B_1 &\leq C_N (t-s)^{2N-1} \int_s^t \mathbb{E}^W \left[|b(k_n(u), X^n(k_n(u)))|^{2N} \right] du \\ &\leq R_N (t-s)^{2N}. \end{aligned}$$

By the Hölder and Burkholder inequalities and using (5.3), we obtain

$$B_2 \leq C_N (t-s)^{N-1} \mathbb{E}^W \left[\int_s^t |\sigma_W(k_n(u), X^n(k_n(u)))|^{2N} du \right] \leq R_N (t-s)^N.$$

Finally, using the estimate (3.4) and the Hölder inequality, we have

$$\left| \int_s^t f(u) dB_u^H \right|^{2N} \leq C_N \Lambda_\alpha(B^H)^{2N} (t-s)^{2N(1-\alpha)+2\alpha-1} \int_s^t \frac{\|f(r)\|_\alpha^{2N}}{(r-s)^{2\alpha}} dr.$$

Applying this estimate, the assumptions (H σ_H) and (5.3), we obtain

$$\begin{aligned} B_3 &\leq \mathbb{E}^W \Lambda_\alpha(B^H)^{2N} (t-s)^{2N(1-\alpha)+2\alpha-1} \int_s^t \frac{\|\sigma_H(k_n(r), X^n(k_n(r)))\|_\alpha^{2N}}{(r-s)^{2\alpha}} dr \\ &\leq R_N (t-s)^{2N(1-\alpha)+2\alpha-1} \mathbb{E}^W \left[\int_s^t \frac{1 + \|X^n(k_n(r))\|_\alpha^{2N}}{(r-s)^{2\alpha}} dr \right] \\ &\leq R_N (t-s)^N, \end{aligned}$$

which concludes the proof. ■

As a consequence of Proposition 5.1, we establish the tightness of the law of the sequence $\{X^n\}_{n \in \mathbb{N}}$ in the space C_0^η of η -Hölder continuous functions, with $\eta < \frac{1}{2}$, such that

$$\lim_{\varepsilon \rightarrow 0} \sup_{0 < |t-s| < \varepsilon} \frac{|f(t) - f(s)|}{(t-s)^\eta} = 0.$$

These spaces are complete and separable [10].

Proposition 5.2 *Let $P^n = P \circ X^n$, $n \geq 0$, be the sequence of probability measures induced by X^n on C_0^η . Then this sequence is tight.*

Proof. Fix $\varepsilon > 0$ and $\eta < \frac{1}{2}$. It suffices to show that there exists a compact set K in C_0^η such that $\sup_{n \geq 0} P(X^n \in K^c) \leq \varepsilon$. Choose an integer N such that $\frac{1}{2} - \frac{1}{2N} > \eta$. Let $M > 0$ be such that

$$P(R_N > M) \leq \frac{\varepsilon}{2}. \quad (5.5)$$

Define a new probability by

$$Q(B) = \frac{P(B \cap \{R_N \leq M\})}{P(R_N \leq M)}.$$

Then, Proposition 5.1 implies that

$$\mathbb{E}_Q \left[|X_t^n - X_s^n|^{2N} \right] = \frac{\mathbb{E} \left[|X_t^n - X_s^n|^{2N} \mathbf{1}_{\{R_N \leq M\}} \right]}{P(R_N \leq M)} \leq M P(R_N \leq M)^{-1} |t-s|^N.$$

By the tightness criterion established in [14], the sequence of probabilities $Q \circ X_n^{-1}$, $n \geq 0$, is tight in C_0^η . Therefore, there exists a compact subset K in C_0^η such that

$$\sup_{n \geq 0} Q(X^n \in K^c) \leq P(R_N \leq M)^{-1} \frac{\varepsilon}{2}. \quad (5.6)$$

Finally, from (5.5) and (5.6) we obtain

$$P(X^n \in K^c) \leq P(X^n \in K^c, R_N \leq M) + P(R_N > M) \leq \varepsilon,$$

which allows us to conclude the proof. ■

Now we can show the existence of a weak solution for Equation (1.2).

Theorem 5.3 *Assume that the coefficients b , σ_W and σ_H satisfy the assumptions (Hb) , $(H\sigma_W)$ and $(H\sigma_H)$. If $1 - H < \alpha < \min \left\{ \frac{1}{2}, \beta, \frac{\delta}{2} \right\}$, then there exists a unique weak solution X of Equation (1.2).*

Proof. The proof will be done in several steps.

Step 1.- By the Prohorov theorem, the sequence $\{P^n, n \geq 0\}$ is weakly relatively compact in C_0^η and exists a subsequence, that we still denote by P^n , which is weakly convergent to some probability P^∞ . By the Skorokhod representation theorem, there exists a sequence of processes $\{Y^n, B^n, W^n, 0 \leq n \leq \infty\}$, defined on some probability space (Ω, \mathcal{F}, P) and with values in C_0^η , such that, for every $0 \leq n \leq \infty$, the process (Y^n, B^n, W^n) has law P^n and

$$\lim_{n \rightarrow \infty} \|Y^n - Y^\infty\|_\eta + \|B^n - B^\infty\|_\eta + \|W^n - W^\infty\|_\eta = 0$$

almost surely.

Since, for every n , the process (Y^n, B^n, W^n) has the same law as (X^n, B^H, W) , if we introduce the filtrations

$$\begin{aligned}\mathcal{F}_t^n &= \sigma \{Y^n(s), B^n(s), W^n(s), s \leq t\}, \\ \mathcal{F}_t^\infty &= \sigma \{Y^\infty(s), B^\infty(s), W^\infty(s), s \leq t\},\end{aligned}$$

the process W^n (resp. W^∞) is an \mathcal{F}_t^n (resp. \mathcal{F}_t^∞) r -dimensional standard Brownian motion. Moreover, B^n and B^∞ are fractional Brownian motions.

Step 2.- By an adaptation of a result in [23] (page 32) or Lemma 3.1 in [9], for any continuous function $f(t, x)$ which satisfies the linear growth property in the variable x , we have that

$$\begin{aligned}\lim_{n \rightarrow \infty} \int_0^t f(k_n(s), Y^n(k_n(s))) ds &= \int_0^t f(s, Y^\infty(s)) ds, \\ \lim_{n \rightarrow \infty} \int_0^t f(k_n(s), Y^n(k_n(s))) dW_s^n &= \int_0^t f(s, Y^\infty(s)) dW_s^\infty,\end{aligned}$$

as n tends to infinity, in probability, and uniformly in $t \in [0, T]$. We have also a similar result for the convergence of integrals with respect to fractional Brownian motions:

$$\lim_{n \rightarrow \infty} \int_0^t \sigma_H(k_n(s), Y^n(k_n(s))) dB_s^n = \int_0^t \sigma_H(s, Y^\infty(s)) dB_s^\infty, \quad (5.7)$$

as n tends to infinity, uniformly in $t \in [0, T]$ and P -a.s. Let us show the convergence (5.7). By the linearity of the generalized Stieltjes integral, it is clear that

$$\left| \int_0^t \sigma_H(k_n(s), Y^n(k_n(s))) dB_s^n - \int_0^t \sigma_H(s, Y^\infty(s)) dB_s^\infty \right| \leq A_1 + A_2,$$

where

$$A_1 = \left| \int_0^t \sigma_H(k_n(s), Y^n(k_n(s))) d(B_s^n - B_s^\infty) \right|$$

and

$$A_2 = \left| \int_0^t [\sigma_H(k_n(s), Y^n(k_n(s))) - \sigma_H(s, Y^\infty(s))] dB_s^\infty \right|.$$

Using the estimate (3.8), we have that

$$\begin{aligned}& \left\| \int_0^t [\sigma_H(s, Y^n(s)) - \sigma_H(s, Y^\infty(s))] dB_s^\infty \right\|_\alpha \leq \\ & \leq C \Lambda_\alpha(B^\infty) \\ & \times \int_0^t \left((t-s)^{-2\alpha} + s^{-\alpha} \right) [(1 + \Delta Y^n(s) + \Delta Y^\infty(s)) \|Y^n(s) - Y^\infty(s)\|_\alpha] ds \\ & \leq C \Lambda_\alpha(B^\infty) \|Y^n - Y^\infty\|_\eta \left(1 + \|Y^n\|_\eta^\delta + \|Y^\infty\|_\eta^\delta \right) \rightarrow 0,\end{aligned}$$

as n tends to infinity, P -a.s. Using the estimate (3.4) and the assumptions in $(H\sigma_H)$, we have

$$\begin{aligned} & \left\| \int_0^\cdot (\sigma_H(k_n(s), Y^n(k_n(s))) - \sigma_H(s, Y^n(s))) dB_s^\infty \right\|_\infty \\ & \leq C \Lambda_\alpha(B^\infty) \|\sigma_H(k_n(s), Y^n(k_n(s))) - \sigma_H(s, Y^n(s))\|_{\alpha,1} \\ & \leq C \Lambda_\alpha(B^\infty) \|\sigma_H(k_n(\cdot), Y^n(k_n(\cdot))) - \sigma_H(\cdot, Y^n(\cdot))\|_\infty^\varepsilon (I_1 + I_2 + I_3), \end{aligned}$$

where $\varepsilon > 0$ is a small positive number that depends on α and β and we have

$$\begin{aligned} I_1 &= \|\sigma_H(k_n(\cdot), Y^n(k_n(\cdot))) - \sigma_H(\cdot, Y^n(\cdot))\|_\infty^{1-\varepsilon}, \\ I_2 &= \int_0^T \int_0^s \frac{|\sigma_H(s, Y^n(s)) - \sigma_H(r, Y^n(r))|^{1-\varepsilon}}{(s-r)^{\alpha+1}} dr ds \\ &\leq C_1 + C_2 \int_0^T \int_0^s \frac{|Y^n(s) - Y^n(r)|^{1-\varepsilon}}{(s-r)^{\alpha+1}} dr ds \\ &\leq C_1 + C_2 \|Y^n\|_\eta^{(1-\varepsilon)\eta} \leq C, \end{aligned}$$

where we have used the Hölder continuity in time in assumption $(H\sigma_H)$, and

$$\begin{aligned} I_3 &= \int_0^T \int_0^s \frac{|\sigma_H(k_n(s), Y^n(k_n(s))) - \sigma_H(k_n(r), Y^n(k_n(r)))|^{1-\varepsilon}}{(s-r)^{\alpha+1}} dr ds \\ &\leq C \int_0^T \int_0^s \frac{|k_n(s) - k_n(r)|^{(1-\varepsilon)\beta} + |k_n(s) - k_n(r)|^{(1-\varepsilon)\eta}}{(s-r)^{\alpha+1}} dr ds \end{aligned}$$

We can compute this last integral, using the partition on the interval and decomposing the integrals in finite sums. This integral is uniformly bounded in n . Therefore

$$\left\| \int_0^\cdot (\sigma_H(k_n(s), Y^n(k_n(s))) - \sigma_H(s, Y^n(s))) dB_s^\infty \right\|_\infty \rightarrow 0,$$

as n tends to infinity, P -a.s. In order to show the convergence of the term A_1 , we use again the estimate (3.4) and Lemmas 7.4 and 7.5 in [19]. We obtain that

$$\begin{aligned} A_1 &\leq \|\sigma_H(k_n(s), Y^n(k_n(s)))\|_{\alpha,1} \Lambda_\alpha(B_s^n - B_s^\infty) \\ &\leq C \|B^n - B^\infty\|_\infty^\varepsilon, \end{aligned}$$

where $\varepsilon > 0$ is a small positive constant which depends on α . Therefore, A_1 converges to zero as n tends to infinity, P -a.s.

Step 3.- Recall from step 1 that (Y^n, B^n, W^n) and (X^n, B^H, W) have the same laws. Moreover, W^n is a standard Brownian motion in the appropriate filtration and B^n is a fractional Brownian motion. Therefore, our processes satisfy the stochastic differential equations

$$\begin{aligned} Y_t^n &= Y_0^n + \int_0^t b(k_n(s), Y^n(k_n(s))) ds \\ &\quad + \int_0^t \sigma_W(k_n(s), Y^n(k_n(s))) dW_s^n + \int_0^t \sigma_H(k_n(s), Y^n(k_n(s))) dB_s^n \end{aligned}$$

almost surely. So, by step 2, when n tends to infinity, we obtain

$$Y_t^\infty = Y_0^\infty + \int_0^t b(s, Y_s^\infty) ds + \int_0^t \sigma_W(s, Y_s^\infty) dW_s^\infty + \int_0^t \sigma_H(s, Y_s^\infty) dB_s^\infty.$$

Therefore, Y^∞ satisfies (1.2) with the driving noises W^∞ and B^∞ .

The sample paths of Y^∞ belong to $C_0^\eta \subset W_0^{\alpha, \infty}$ almost surely, and furthermore, by (5.3), we have that

$$\int_0^T \mathbb{E}^W \left[\|Y_s^\infty\|_\alpha^2 \right] ds < \infty.$$

Therefore, by Definition 4.1, $(Y^\infty, W^\infty, B^\infty)$ is a weak solution of (1.2). ■

We can now proceed with the proof of Theorem 2.2.

Proof of Theorem 2.2. The uniqueness is a consequence of the general pathwise uniqueness proved in Theorem 4.4. For the existence of a strong solution we can make use of the classical result by Yamada and Watanabe [24], which asserts that pathwise uniqueness and existence of weak solutions imply the existence of a strong solution. The main difference with the classical proof is that here we have two random sources independent of the Wiener process W , the initial condition X_0 and the fractional Brownian motion B^H . It suffices to replace \mathbb{R}^d by the product space $\mathbb{R}^d \times C([0, T])^m$, endowed with the product measure $\mu \times \nu$, where μ is the law of X_0 and ν is the law of B^H on the space of continuous functions. ■

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